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Quantum deformed Poincaré algebra on a two-dimensional lattice

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Abstract. We propose a definition of a Poincaré algebra for a two-dimensional spacetime with one discretized dimension. This algebra has the structure of a Hopf algebra. We use the link between Onsager's uniformization of the Ising model and the dispersion relation of a free particle in this spacetime, together with the rapidity representation of the quantum deformation of the Poincaré enveloping algebra.

1. Lattice Poincaré algebra

Lattice generalizations of Poincaré or inhomogeneous Lorentz invariance have been proposed in the context of two-dimensional integrable models [1, 2]. These works were inspired by Onsager's solution of the Ising model [3] and Baxter's definition of the corner transfer matrix [4].

Onsager's solution to the Ising model provides a natural way of associating a continuous rapidity to a free massive fermion positioned in a discrete two-dimensional spacetime. In fact, mapping the lattice spacing a_x, a_t [2] into the Ising couplings H and H', H^* being the dual of H [3],

$$\sinh 2H' \sinh 2H^* = \left(\frac{a_x}{a_t}\right)^2 \qquad 2\sinh(H' - H^*) = \mu a_x \tag{1.1}$$

and defining

$$\gamma = pa_x \qquad \omega = Ea_t \tag{1.2}$$

we get, from Onsager's hypergeometric relation,

$$\cosh\gamma = \cosh 2H' \cosh 2H^* - \sinh 2H' \sinh 2H^* \cos\omega \tag{1.3}$$

the lattice dispersion relation

$$a_t^2(\cosh pa_x - 1) + a_x^2(\cos Ea_t - 1) = \tfrac{1}{2}\mu^2 a_t^2 a_x^2. \tag{1.4}$$

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Now, using Onsager's uniformization of equation (1.3), we get the desired rapidity for free particles moving in a discrete two-dimensional spacetime. For fixed Ising couplings H, H', the Onsager parameters α and u are defined by the following uniformization relations in terms of Jacobi elliptic functions:

$$\begin{aligned}
\sinh 2H' &= -\mathrm{i}\,\mathrm{sn}(\mathrm{i}u|k^2) \\
\sinh 2H^* &= -\mathrm{i}k\,\mathrm{sn}(\mathrm{i}u|k^2) \\
\sinh\gamma &= -\mathrm{i}\frac{1-k^2}{M}\,\mathrm{sn}(\mathrm{i}u|k^2) \\
\sin\omega &= \frac{1-k^2}{M}\,\mathrm{sn}(\alpha|k^2)
\end{aligned} \tag{1.5}$$

with the notation

$$M = \mathrm{dn}(\mathrm{i}u|k^2)\,\mathrm{dn}(\alpha|k^2) + k\,\mathrm{cn}(\mathrm{i}u|k^2)\,\mathrm{cn}(\alpha|k^2). \tag{1.6}$$

The elliptic modulus k is defined by the Ising integrability relation

$$k = \frac{\sinh 2H^*}{\sinh 2H'} \tag{1.7}$$

and for $H' > H^*$, $k < 1$. As usual, integrability means the commutativity of the transfer matrix for two different values of the parameter α, which turns out to be the rapidity.

To show this, we introduce the substitution (1.1) into (1.5) and obtain the following relation for the elliptic variables u and k in terms of the lattice variables a_x, a_t:

$$\begin{aligned}
-\mathrm{i}\,\mathrm{sn}(\mathrm{i}u|k^2) &= \frac{1}{\sqrt{k}}\frac{a_x}{a_t} \\
M &= \frac{\sqrt{k}}{a_t}\left(\sqrt{\frac{ka_x^2+a_t^2}{k}}\,\mathrm{dn}(\alpha|k^2) + \sqrt{a_x^2+ka_t^2}\,\mathrm{cn}(\alpha|k^2)\right) \\
k_\pm &= 1 + A \pm \sqrt{2A + A^2} \\
A &= \tfrac{1}{2}\mu^2(a_t^2+a_x^2) + \tfrac{1}{8}\mu^4 a_t^2 a_x^2.
\end{aligned} \tag{1.8}$$

In the continuum limit where both $a_x, a_t \to 0$, and therefore $k \to 1$, equations (1.2), together with (1.5), become

$$p = \mu\cosh\alpha \qquad E = \mu\sinh\alpha \tag{1.9}$$

which shows that α is the rapidity, with the mass shell condition $p^2 - E^2 = \mu^2$ (although E and p are usually interchanged). This suggests that we define a boost generator N by

$$N = \frac{\partial}{\partial\alpha}. \tag{1.10}$$

Relations (1.9) can be obtained as solutions to the differential equations

$$\frac{\partial P(\alpha)}{\partial\alpha} = E(\alpha) \qquad \frac{\partial^2 P(\alpha)}{\partial\alpha^2} = P(\alpha) \tag{1.11}$$

which are equivalent to the standard two-dimensional Poincaré algebra in the continuum

$$[N, P] = E \qquad [N, E] = P \qquad [E, P] = 0 \tag{1.12}$$

once we use the rapidity representation (1.10) for the boost generator.

In order to define the lattice generalization of the two-dimensional Poincaré algebra, we should modify algebra (1.12) in such a way that the lattice rapidity representation obtained using Onsager's uniformization (1.5) now appears as a solution to the differential equations defined by the modified algebra.

In this paper we prove that, at least for a discrete space and a continuous time ($a_t \to 0$), the lattice generalization of (1.12) is a Hopf algebra possessing an asymmetric comultiplication for N and E.

2. From lattice Poincaré to quantum Poincaré algebra

A quantum deformation of the Poincaré algebra was introduced in [5, 6]. For two dimensions, the κ-Poincaré algebra $U_\kappa(P_2)$ is defined by the commutation relations

$$[N, P] = E \qquad [N, E] = \kappa \sinh \frac{P}{\kappa} \qquad [E, P] = 0 \tag{2.1}$$

with $\kappa \in \mathbb{R}$. Introducing a rapidity η, we obtain, from (2.1),

$$\frac{\partial P(\eta)}{\partial \eta} = E(\eta) \qquad \frac{\partial^2 P(\eta)}{\partial \eta^2} = \kappa \sinh \frac{P(\eta)}{\kappa}. \tag{2.2}$$

It is now easy to prove that the semi-continuous limit $a_t \to 0$ of equations (1.5) defines a solution to equations (2.2). In fact, solving (2.2), we obtain [7]

$$2\kappa \sinh \frac{P(\eta)}{2\kappa} = \mu \operatorname{nc}(K_L^{-1}\eta | K_L^2) \qquad E(\eta) = \mu \operatorname{sc}(K_L^{-1}\eta | K_L^2) \tag{2.3}$$

with elliptic modulus

$$K_L^2 = \frac{1}{1 + \mu^2/4\kappa^2}. \tag{2.4}$$

On the other hand, taking the $a_t \to 0$ limit of (1.5), we obtain

$$\sinh \gamma = \frac{1-k^2}{k} \frac{1}{\operatorname{cn}\alpha + \operatorname{dn}\alpha} \qquad \sin \omega \simeq E a_t = \frac{a_t}{a_x} \frac{1-k^2}{k} \frac{\operatorname{sn}\alpha}{\operatorname{cn}\alpha + \operatorname{dn}\alpha} \tag{2.5}$$

whereby

$$E = \frac{1}{a_x} \frac{1-k^2}{k} \frac{\operatorname{sn}\alpha}{\operatorname{cn}\alpha + \operatorname{dn}\alpha}. \tag{2.6}$$

By performing a Landen transformation [8] we can identify (2.3) and (2.5), provided

$$\sqrt{k}\alpha = \eta \qquad \text{and} \qquad a_x = \frac{1}{\kappa}. \tag{2.7}$$

Therefore, we conclude that in the semi-continuous limit, the two-dimensional lattice Poincaré algebra is defined by quantum deformation (2.1), with the deformation parameter κ being determined by the lattice spacing. As an extra piece of evidence, notice that lattice dispersion relation (1.4) becomes, in the $a_t \to 0$ limit, the Casimir operator

$$C = \left(2\kappa \sinh \frac{p}{2\kappa}\right)^2 - E^2 = \mu^2 \tag{2.8}$$

of the quantum Poincaré algebra (2.1).

In the continuum limit $a_x \to 0$ (i.e. $\kappa \to \infty$), the elliptic parameter k becomes 1 (see (1.8)), which implies $H' = H^*$ in (1.7), i.e. $T = T_c$. Hence κ measures the departure from criticality.

Algebra (2.1) can be promoted to a Hopf algebra by the following comultiplication rules:

$$\begin{aligned} \Delta P &= P \otimes 1 + 1 \otimes P \\ \Delta E &= E \otimes e^{P/2\kappa} + e^{-P/2\kappa} \otimes E \\ \Delta N &= N \otimes e^{P/2\kappa} + e^{-P/2\kappa} \otimes N \end{aligned} \tag{2.9}$$

and antipode

$$S(P) = -P \qquad S(E) = -E \qquad S(N) = -N + E/2\kappa \tag{2.10}$$

as well as a trivial co-unit $\epsilon(X) = 0$ for $X = P, E, N$. An interesting feature is that, for finite lattice spacing, the comultiplications for E and N are asymmetric.

According to (2.9), the total energy and momentum for a system of two particles is given by

$$P^t = P_1 + P_2 \qquad E^t = E_1 e^{P_2/2\kappa} + e^{-P_1/2\kappa} E_2 \tag{2.11}$$

satisfying the relation following from Casimir operator (2.8)

$$\mathrm{cst} = \left(2\kappa \sinh \frac{P^t}{2\kappa}\right)^2 - (E^t)^2. \tag{2.12}$$

It should be noted that the comultiplication (2.9) is not unique. Algebra (2.1) is compatible with a symmetric comultiplication which is additive for E, non-additive for P and complicated for N [9].

3. Final comments

The main goal of this paper is to provide a definition of the lattice Poincaré algebra in terms of well defined differential operators. The main tool that we have used, inspired by integrable models, is the introduction of a continuous but elliptic rapidity that will translate the lattice information into an elliptic parameter. More precisely, the steps followed in defining this algebra were:

(i) the uniformization of the lattice dispersion relation, for a free massive particle, by means of Onsager's uniformization of the Ising model (the Ising model was used simply because of its equivalence to a free fermionic system);

(ii) the representation of the boost generator of the Poincaré algebra as a derivative with respect to the rapidity variable, which is identified with the elliptic uniformization parameter; and

(iii) the integration, using the elliptic rapidity, of the lattice dispersion relation (1.4) in the semi-continuum limit. In this way, we arrive at equations (2.2), which define the quantum Poincaré algebra introduced in [5, 6]. In this algebra, the only memory of its lattice origin is relation (2.7) between the quantum deformation parameter and the lattice spacing.

The unexpected outcome of this exercise, which was performed on the basis of one-body information, is the kinematical implication for the many-body dynamics which is encoded in the Hopf algebra structure (comultiplication rules) of the quantum (lattice) Poincaré algebra. The non-triviality of the comultiplication rules directly derives from the non-trivial topology of the rapidity space, i.e. we have lost two conformal Killing vectors. The situation is somewhat similar to the standard analogy between quantum $SU(2)$ and the asymmetric top.

The physical meaning of the comultiplication rules becomes more complicated when dealing with a free system. In general, in order to define the free Hamiltonian for n particles, we just add n free one-body Hamiltonians, which is equivalent to assuming trivial comultiplication rules for kinematical observables. Therefore, we observe that the quantum deformation of any kinematical symmetry requires the existence of some non-local interactions to account for the non-trivial comultiplication.

There are two physical questions that deserve some attention at this point. The first concerns the interplay between integrability and lattice kinematics, interpreted in the way proposed in this paper. For an integrable model in the elliptic regime, we can always try to define an associated Poincaré algebra where the rapidity is the uniformization parameter and the boost generator corresponds to the corner transfer matrix. From this kinematical point of view, integrability becomes equivalent to the relativity principle, i.e. the physics is the same for two observers related by a boost, which corresponds to the commutativity of the transfer matrices for two different values of the uniformization parameter. The Poincaré algebra so obtained is very likely to be quantum deformed with the quantum deformation parameter being determined by the integrability constants (the parameters determining the integrability manifold in the space of couplings) of the model. It would be very interesting, if the previous picture is correct, to find the physical interpretation of the comultiplication rules in this context. It should be noted that a lattice Poincaré algebra was previously defined [1, 2] for integrable models using an infinite set of conserved charges. In the lattice Poincaré algebra considered here, we play with only three generators but are forced to work, as is customary for quantum deformations, with the enveloping algebra.

The second question concerns the interpretation of the quantum Poincaré algebra as a lattice regularization. From the two-dimensional semi-continuous case studied here, we learn that the lattice spacing transmutes into a quantum deformation parameter, automatically modifying the comultiplication rules. A possible physical picture for understanding this phenomenon is to interpret the non-trivial comultiplication rules as a way of formally saving the conservation laws lost when postulating a fundamental length. An interesting exercise in this direction would be to derive, from the deformed Poincaré algebra and for two-dimensional models, the germ (i.e. the Möbius part) of a deformed conformal algebra useful away from criticality.

To conclude, we wish to point out that perhaps the most pragmatic and useful approach to quantum deformed kinematics is to interpret it as a subtle form of lattice regularization.

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